

SYMMETRIZED TRACE AND SYMMETRIZED DETERMINANT OF ODD CLASS PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We introduce a new canonical trace on odd class logarithmic pseudo-differential operators on an odd dimensional manifold, which vanishes on commutators. When restricted to the algebra of odd class classical pseudo-differential operators our trace coincides with the canonical trace of Kontsevich and Vishik. Using the new trace we construct a new determinant of odd class classical elliptic pseudo-differential operators. This determinant is multiplicative up to sign whenever the multiplicative anomaly formula for usual determinants of Kontsevich-Vishik and Okikiolu holds. When restricted to operators of Dirac type our determinant provides a sign refined version of the determinant constructed by Kontsevich and Vishik. We discuss some applications of the symmetrized determinant to a non-linear σ -model in superconductivity.

1. INTRODUCTION

To define a trace and a determinant of an unbounded (pseudo-)differential operator one needs to use a regularization procedure, which usually leads to so called anomalies in the behavior of the obtained regularized trace and determinant. Namely

- The obtained trace and determinant depend on the choices made for the regularization.
- The regularized trace is not tracial, i.e., does not vanish on commutators.
- The regularized determinant is not multiplicative, i.e., $\text{Det } AB \neq \text{Det } A \cdot \text{Det } B$.

It is well known that, in general, those anomalies are unavoidable. For example, there is no extension of the usual trace of trace class operators to the algebra of all pseudo-differential operators, which is tracial. However, if one restricts to a subalgebra one can hope to have a tracial extension of the usual trace, cf. [9]. In particular, Kontsevich and Vishik [12, §4] constructed a tracial functional $\text{Tr}_{(-1)}$ on the algebra of odd class classical pseudo-differential operators on an odd dimensional manifold.

The usual regularization procedure depends on the choice of a spectral cut of the complex plane. In this paper we propose a new regularized trace and determinant, called the *symmetrized trace* and the *symmetrized determinant*, of odd class operators on an odd dimensional manifold, obtained by averaging the usual definitions for different spectral cuts. The symmetrized trace is independent of any choices. The symmetrized determinant still depends on the spectral cut, but this dependence is in a sense weaker than for the usual regularized trace and determinant. The symmetrized trace is tracial and the symmetrized determinant is very often multiplicative. We will now explain our constructions in some more details.

1.1. The symmetrized trace. Fix an elliptic pseudo-differential operator Q of a positive order m and assume that θ is an Agmon angle for Q , cf. Subsection 2.6. The Q -weighted trace $\text{Tr}_{(\theta)}^Q A$

of a logarithmic pseudo-differential operator A is defined to be the constant term in the Laurent expansion at $s = 0$ of the function $s \mapsto \text{TR}(AQ_{(\theta)}^s)$. Here, $Q_{(\theta)}^s$ is the s -th power of Q defined using the spectral cut θ and TR is the Kontsevich-Vishik canonical trace, introduced in [13, §3], [12, §3] for classical pseudo-differential operators, and generalized in [14, §5]. The weighted trace depends on both, Q and θ , and, in general, is not tracial.

Suppose now that the dimension of the manifold is odd, that both operators Q and A are of odd class, cf. Section 2, and that both, θ and $\theta - m\pi$, are Agmon angles for Q . Then we define the *symmetrized trace* $\text{Tr}^{\text{sym}} A$ of A by the formula

$$\text{Tr}^{\text{sym}} A := \frac{1}{2} \left(\text{Tr}_{(\theta)}^Q A + \text{Tr}_{(\theta-m\pi)}^Q A \right).$$

Then, cf. Proposition 3.7, the symmetrized trace is independent of the choice of Q and θ . If A is a classical pseudo-differential operator, then $\text{Tr}^{\text{sym}} A$ coincides with the Kontsevich-Vishik canonical trace $\text{Tr}_{(-1)} A$, cf. Subsection 3.10. Recall that Kontsevich and Vishik defined $\text{Tr}_{(-1)} A$ using an even order positive definite operator Q . Thus we also obtain a new formula for $\text{Tr}_{(-1)} A$, which defines it using an operator which is not necessarily positive definite.

To define a determinant of A , one needs to define a trace of the logarithm of A , which is not a classical pseudo-differential operator. Hence, we are mostly interested in the properties of the symmetrized trace on the space of logarithmic pseudo-differential operators. Our main result here is Theorem 3.9, which claims that Tr^{sym} is tracial in the sense that

$$\text{Tr}^{\text{sym}}[A, B] = 0,$$

for any odd class logarithmic pseudo-differential operators A and B .

1.2. The symmetrized determinant. Assume now that A is an odd class elliptic classical pseudo-differential operator of positive integer order m and that both, θ and $\theta - m\pi$, are Agmon angles for A . We define the *symmetrized determinant* $\text{Det}_{(\theta)}^{\text{sym}} A$ of A by the formula

$$\log \text{Det}_{(\theta)}^{\text{sym}} A := \frac{1}{2} \text{Tr}^{\text{sym}} (\log_{(\theta)} A + \log_{(\theta-m\pi)} A). \quad (1.1)$$

If the order m of A is even, then the symmetrized determinant (1.1) is equal to the usual ζ -regularized determinant of A , cf. Proposition 4.4, but in the case when m is odd, the symmetrized determinant might be quite different from the usual one. By Proposition 4.5, in this case

$$(\text{Det}_{(\theta)}^{\text{sym}} A)^2 = \text{Det}_{(2\theta)}^{\text{sym}}(A^2). \quad (1.2)$$

Combining Propositions 4.4 and 4.5 we see that up to sign the symmetrized determinant is quite a classical object. Definition (1.1) fixes a sign which is crucial for applications in Subsections 1.3 and 1.4. Also our proof of the multiplicativity of the symmetrized determinant relies heavily on the construction of the symmetrized trace and formula (1.1).

As the usual ζ -regularized determinant, the symmetrized determinant does depend on the spectral cut θ , but this dependence is weaker than in the case of the usual determinant, cf. Proposition 4.7. In particular, if the order of A is odd and its leading symbol is self-adjoint, then $\text{Det}_{(\theta)}^{\text{sym}} A$ is independent of θ up to sign.

Our main result is Theorem 5.1. Roughly speaking it says that whenever the multiplicative anomaly formula of Kontsevich-Vishik and Okikiolu for usual determinants holds, the symmetrized determinant is multiplicative up to sign. For example, cf. Corollary 5.3, if the leading symbols of the operators A and B commute and the leading symbol of A is self-adjoint, then

$$\text{Det}_{(\theta_B + \varepsilon \theta_A)}^{\text{sym}} AB = \pm \text{Det}_{(\theta_A)}^{\text{sym}} A \cdot \text{Det}_{(\theta_B)}^{\text{sym}} B,$$

where $\theta_A, \theta_B \in (0, 2\pi)$ are Agmon angles for the operators A and B respectively and $\varepsilon = 1$ if $\theta_A \in (0, \pi)$ and $\varepsilon = -1$ if $\theta_A \in (\pi, 2\pi)$.

In the case when the operator A is of Dirac type Kontsevich and Vishik, [12, §4.1], suggested to define a new determinant of A as a square root of the determinant of the Laplace-type operator A^2 . Thus their definition had a sign indeterminacy. It follows from the equation (1.2) that the reduction modulo signs of the symmetrized determinant $\text{Det}_{(\theta)}^{\text{sym}} A$ coincides in the case of Dirac-type operators with the Kontsevich-Vishik determinant. Thus in the case of Dirac-type operators our symmetrized determinant provides a more direct construction of the Kontsevich-Vishik determinant and also fixes the signs in its definition.

1.3. Operators with spectrum symmetric about the real axis. Let A be an odd class classical pseudo-differential operator of odd order. Assume that the leading symbol of A is self-adjoint with respect to a Hermitian metric on A . Suppose further that the spectrum of A is symmetric about the real axis, cf. Definition 6.1. Note that any differential operator with real coefficients has this property. We denote by m_+ the number of the eigenvalues of A (counting with their algebraic multiplicities) which lie on the positive part of the imaginary axis (since the spectrum of A is symmetric m_+ is equal to the number m_- of the eigenvalues which lie on the negative part of the imaginary axis). In Theorem 6.2 we show that *the symmetrized determinant $\text{Det}_{(\theta)}^{\text{sym}} A$ is real and its sign is equal to $(-1)^{m_+}$, i.e.,*

$$\text{Det}_{(\theta)}^{\text{sym}} A = (-1)^{m_+} \left| \text{Det}_{(\theta)}^{\text{sym}} A \right|. \quad (1.3)$$

Note that this result is somewhat surprising, since for a finite dimensional matrix whose spectrum is symmetric about the real axis the sign of the determinant is independent of the number of imaginary eigenvalues and is determined by the number of eigenvalues which lie on the negative part of the real axis.

The equation (1.3) should be compared with the main theorem of [1] where a similar equality was obtained for the usual ζ -regularized determinant of an operator whose spectrum is symmetric about the imaginary axis.

1.4. Application to a non-linear σ -model. In the recent years several examples appeared in physical literature when the phase of the determinant of a geometrically defined non self-adjoint Dirac-type operator is a topological invariant, see e.g., [18, 19, 5, 3, 4, 2]. In some of these examples the physicists claim that the determinant is real and compute its sign. Unfortunately, their arguments are not rigorous. In particular, they never specify which spectral cut they use to define the determinant. In [1] we tried to better understand this phenomenon. In particular, we provided a rigorous computation of the sign of roughly one half of the examples

from [4, 2] in which the spectrum of the Dirac-type operator is symmetric about the imaginary axis. Unfortunately, the methods of [1] are not applicable to the other half of the examples in which the spectrum is symmetric about the real axis, cf. the operator $\Gamma \circ D_{mn}$ in section 6.3 of [1]. Moreover, one can show that the usual ζ -regularized determinant is not real in some of these examples and the computations of physicists are not legal, essentially because they don't take into account the effect of the choice of the spectral cut.¹ There is, however, no reason to believe that the usual ζ -function regularization of the determinant is the most adequate for the description of physics. Moreover, it turns out that the σ -model constructed using the usual ζ -regularized determinant fails to satisfy some basic conservation laws. One of our main motivation for the study of the symmetrized determinants was an attempt to give a rigorous description of the results of [4, 2]. We will discuss the obtained σ -models elsewhere. In particular, we will explain that (1.3) gives the sign of the determinant predicted in [4, 2].

2. PRELIMINARIES

2.1. Classical pseudo-differential operators of odd class. Let M be a closed d -dimensional manifold and let E be a complex vector bundle over M . We denote by $\text{CL}^m(M, E)$ the space of order $m \in \mathbb{C}$ classical pseudo-differential operators

$$A : C^\infty(M, E) \longrightarrow C^\infty(M, E),$$

cf. [21]. Recall that for each $A \in \text{CL}^m(M, E)$ the symbol $\sigma(A)(x, \xi)$ has an asymptotic expansion of the form

$$\sigma(A)(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(A)(x, \xi), \quad (x, \xi) \in T^*M, \quad (2.1)$$

where each $\sigma_{m-j}(x, \xi)$ is *positive homogeneous* in ξ of degree $m - j$, i.e.

$$\sigma_{m-j}(A)(x, t\xi) = t^{m-j} \sigma_{m-j}(A)(x, \xi), \quad \text{for every } t > 0. \quad (2.2)$$

The functions $\sigma_{m-j}(A)(x, \xi)$ are called the *positive homogeneous components* of the symbol of A . The function $\sigma_m(x, \xi)$ is called the *principal symbol* of A .

Set $\text{CL}(M, E) = \bigcup_{m \in \mathbb{C}} \text{CL}^m(M, E)$.

An *odd class* operator of an *integer* order m is an operator $A \in \text{CL}^m(M, E)$ such that

$$\sigma_{m-j}(A)(x, -\xi) = (-1)^{m-j} \sigma_{m-j}(A)(x, \xi), \quad j = 0, 1, \dots \quad (2.3)$$

We denote by $\text{CL}_{(-1)}^m(M, E)$ the space of odd class operators of order $m \in \mathbb{Z}$ and we set

$$\text{CL}_{(-1)}(M, E) = \bigcup_{m \in \mathbb{Z}} \text{CL}_{(-1)}^m(M, E). \quad (2.4)$$

Note that all *differential* operators belong to $\text{CL}_{(-1)}(M, E)$.

¹More precisely, they consider a deformation $A(t)$ of A and don't take into account the fact that one can not take the same spectral cut for all values of t .

2.2. Logarithmic differential operators. We say that a pseudo-differential operator $A : C_c^\infty(M, E) \rightarrow C^\infty(M, E)$ is *logarithmic* of degree $m \in \mathbb{C}$, cf. [16, 17], if its symbol has an asymptotic expansion of the form

$$\sigma(A)(x, \xi) \sim \gamma \log |\xi| + \sum_{j=0}^{\infty} \sigma_{m-j}(A)(x, \xi), \quad (x, \xi) \in T^*M, \quad m, \gamma \in \mathbb{C}, \quad (2.5)$$

where each $\sigma_{m-j}(A)(x, \xi)$ is positive homogeneous in ξ of degree $m - j$. The number $m \in \mathbb{C}$ is called the degree of the logarithmic pseudo-differential operator A , and the number $\gamma \in \mathbb{C}$ is called the *type* of this operator. We denote the set of logarithmic pseudo-differential operators of degree m and type γ ((m, γ) -logarithmic, for short) by $\widetilde{\text{CL}}^{m, \gamma}(M, E)$ and we set

$$\widetilde{\text{CL}}^m(M, E) = \bigcup_{\gamma \in \mathbb{C}} \widetilde{\text{CL}}^{m, \gamma}(M, E), \quad \widetilde{\text{CL}}(M, E) = \bigcup_{m \in \mathbb{Z}} \widetilde{\text{CL}}^m(M, E). \quad (2.6)$$

Note that logarithmic pseudo-differential operators are not classical.

We say that a logarithmic pseudo-differential operator $A \in \widetilde{\text{CL}}^m(M, E)$ ($m \in \mathbb{Z}$) is of *odd class* if in the asymptotic expansion (2.5) each term $\sigma_{m-j}(A)(x, \xi)$ satisfies (2.3). Denote by $\widetilde{\text{CL}}_{(-1)}(M, E)$ (resp. $\widetilde{\text{CL}}_{(-1)}^{m, \gamma}(M, E)$) the set of odd class logarithmic (resp. odd class (m, γ) -logarithmic) pseudo-differential operators.

Using the standard rules of composition of symbols (cf., for example, [21, Th. 3.5], [17, §1]) we immediately get

Lemma 2.3. (i) If $A, B \in \widetilde{\text{CL}}(M, E)$, then the commutator $[A, B] = AB - BA$ is a classical pseudo-differential operator, $[A, B] \in \text{CL}(M, E)$.

(ii) If $A, B \in \widetilde{\text{CL}}_{(-1)}(M, E)$, then $[A, B] \in \text{CL}_{(-1)}(M, E)$.

2.4. Odd pair of logarithmic pseudo-differential operators. We say that the operators $A, B \in \widetilde{\text{CL}}^m(M, E)$ form an *odd pair* (or that the pair (A, B) is odd) if, for all $j = 0, 1, \dots$,

$$\sigma_{m-j}(A)(x, -\xi) = (-1)^{m-j} \sigma_{m-j}(B)(x, \xi), \quad (2.7)$$

where $\sigma_{m-j}(A)(x, \xi)$, $\sigma_{m-j}(B)(x, \xi)$ are positive homogeneous components of the symbols of A and B respectively, cf. (2.5).

Clearly, if $A \in \widetilde{\text{CL}}_{(-1)}(M, E)$ then the pair (A, A) is odd. Further, if (A, B) is an odd pair, then $A + B \in \widetilde{\text{CL}}_{(-1)}(M, E)$.

From the standard formulae of composition of symbols (cf., for example, [21, Th. 3.5], [17, §1]) we immediately obtain

Lemma 2.5. Given two odd pairs (A_1, B_1) and (A_2, B_2) of classical pseudo-differential operators, the pair $(A_1 A_2, B_1 B_2)$ is also odd.

2.6. Complex powers of elliptic pseudo-differential operators. An angle $\theta \in [0, 2\pi)$ is said to be a *principal angle* for a pseudo-differential operator $A \in \text{CL}^m(M, E)$ ($m \in \mathbb{Z}_+$) if there exists a conical neighborhood Λ of the ray

$$R_\theta := \{ \rho e^{i\theta} : \rho \geq 0 \} \quad (2.8)$$

such that, for each $\xi \neq 0$, the principal symbol $\sigma_m(A)(x, \xi)$ of A does not have eigenvalues in Λ . In particular, the existence of a principal angle implies that the operator A is elliptic and that the spectrum of A is discrete.

An *Agmon angle* for A is a principal angle such that there are no eigenvalues of A on the ray R_θ . If A admits an Agmon angle then it is elliptic and invertible. In this situation there exists $\varepsilon > 0$ such that there are no eigenvalues of A in the solid angle

$$\Lambda_{[\theta-\varepsilon, \theta+\varepsilon]} := \{ \rho e^{i\alpha} : \rho \geq 0, \theta - \varepsilon \leq \alpha \leq \theta + \varepsilon \}.$$

Suppose $A \in \text{CL}^m(M, E)$ ($m \in \mathbb{Z}_+$) admits an Agmon angle θ . Then, cf. [20], [21, Ch. II, §8], for each $s \in \mathbb{C}$ a complex power $A_{(\theta)}^s \in \text{CL}^{sm}(M, E)$ is defined. The symbol of $A_{(\theta)}^s$ can be calculated as follows: Let $R(\lambda) = (A - \lambda)^{-1}$ (where $\lambda \in \Lambda_{[\theta-\varepsilon, \theta+\varepsilon]}$) denote the resolvent of A . The symbol $r(x, \xi; \lambda)$ of $R(\lambda)$ has an asymptotic expansion of the form

$$r(x, \xi; \lambda) \sim \sum_{j=0}^{\infty} r_{-m-j}(x, \xi; \lambda), \quad (2.9)$$

where the terms $r_k(x, \xi; \lambda)$ are positive homogeneous in the sense that, for $t > 0$

$$r_k(x, t\xi; t^m \lambda) = t^k r_k(x, \xi; \lambda), \quad (x, \xi) \in T^*M, \lambda \in \Lambda_{[\theta-\varepsilon, \theta+\varepsilon]}. \quad (2.10)$$

Moreover, if $A \in \text{CL}_{(-1)}^m(M, E)$, then it follows from the explicit formulae for $r_k(x, \xi; \lambda)$, cf. [21, §11.1], [12, §2], that

$$r_k(x, -\xi; (-1)^m \lambda) = (-1)^k r_k(x, \xi; \lambda), \quad (x, \xi) \in T^*M, \lambda \in \Lambda_{[\theta-\varepsilon, \theta+\varepsilon]}. \quad (2.11)$$

The symbol $\sigma(A_{(\theta)}^s)(x, \xi)$ has the asymptotic expansion, cf. [21, §11.2], [12, §2],

$$\sigma(A_{(\theta)}^s)(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{sm-j}(A_{(\theta)}^s)(x, \xi), \quad (2.12)$$

where the terms $\sigma_{sm-j}(A_{(\theta)}^s)(x, \xi)$ are positive homogeneous of degree $sm - j$. They depend analytically on $s \in \mathbb{C}$ and for $\text{Re } s < 0$ are given by

$$\sigma_{sm-j}(A_{(\theta)}^s)(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_{(\theta)}} \lambda_{(\theta)}^s r_{-m-j}(x, \xi; \lambda) d\lambda, \quad (2.13)$$

where the contour $\Gamma_{(\theta)} = \Gamma_{(\theta), \rho_0} \subset \mathbb{C}$ consisting of three curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$,

$$\begin{aligned} \Gamma_1 &= \{ \rho e^{i\theta} : \infty > \rho \geq \rho_0 \}, \quad \Gamma_2 = \{ \rho_0 e^{i\alpha} : \theta > \alpha > \theta - 2\pi \}, \\ \Gamma_3 &= \{ \rho e^{i(\theta-2\pi)} : \rho_0 \leq \rho < \infty \}. \end{aligned} \quad (2.14)$$

Here $\rho_0 > 0$ is a small enough number, and $\lambda_{(\theta)}^s$ is defined as $e^{s \log_{(\theta)} \lambda}$, where $\theta \geq \text{Im } \log_{(\theta)} \lambda \geq \theta - 2\pi$ (i.e., $\text{Im } \log_{(\theta)} \lambda = \theta$ on Γ_1 and $\text{Im } \log_{(\theta)} \lambda = \theta - 2\pi$ on Γ_3).

Lemma 2.7. *Suppose $A \in \text{CL}_{(-1)}^m(M, E)$ and that both, θ and $\theta - m\pi$, are Agmon angles for A . Then, for each $j = 0, 1, \dots$, $(x, \xi) \in T^*M$, $s \in \mathbb{C}$,*

$$\sigma_{sm-j}(A_{(\theta)}^s)(x, -\xi) = (-1)^j e^{ims\pi} \sigma_{sm-j}(A_{(\theta-m\pi)}^s)(x, \xi). \quad (2.15)$$

Proof. Assume, first, that $\operatorname{Re} s < 0$. Then, using (2.11), we obtain

$$\begin{aligned}
\sigma_{sm-j}(A_{(\theta)}^s)(x, -\xi) &= \frac{i}{2\pi} \int_{\Gamma_{(\theta)}} \lambda_{(\theta)}^s r_{-m-j}(x, -\xi; \lambda) d\lambda \\
&= (-1)^{m+j} \frac{i}{2\pi} \int_{\Gamma_{(\theta)}} \lambda_{(\theta)}^s r_{-m-j}(x, \xi; (-1)^m \lambda) d\lambda \\
&= (-1)^{m+j} \frac{i}{2\pi} \int_{\Gamma_{(\theta-m\pi)}} (e^{im\pi} \mu)_{(\theta)}^s r_{-m-j}(x, \xi; \mu) d(e^{im\pi} \mu) \\
&= (-1)^{m+j} e^{im\pi} \frac{i}{2\pi} \int_{\Gamma_{(\theta-m\pi)}} e^{ims\pi} \mu_{(\theta-m\pi)}^s r_{-m-j}(x, \xi; \mu) d\mu \\
&= (-1)^j e^{ims\pi} \sigma_{sm-j}(A_{(\theta-m\pi)}^s)(x, \xi).
\end{aligned}$$

Since both, the left and the right hand side of this equality, are analytic in s , we conclude that the equality holds for all $s \in \mathbb{C}$. \square

2.8. Logarithms of elliptic pseudo-differential operators. For $A \in \operatorname{CL}^m(M, E)$ admitting an Agmon angle θ , the logarithm $\log_{(\theta)} A$ of A is defined by the formula

$$\log_{(\theta)} A = \left. \frac{\partial}{\partial s} \right|_{s=0} A_{(\theta)}^s. \quad (2.16)$$

Then, cf. [12, §2], [16, Lemma 2.4], $\log_{(\theta)} A \in \widetilde{\operatorname{CL}}^{0,m}(M, E)$. In particular, the symbol of $\log_{(\theta)} A$ admits an asymptotic expansion

$$\sigma(\log_{(\theta)} A)(x, \xi) \sim m \log |\xi| + \sum_{j=0}^{\infty} \sigma_{-j}(\log_{(\theta)} A)(x, \xi), \quad (2.17)$$

and its positive homogeneous components $\sigma_{-j}(\log_{(\theta)} A)(x, \xi)$ are given by the formulae

$$\sigma_{-j}(\log_{(\theta)} A)(x, \xi) = |\xi|^{-j} \partial_s \sigma_{sm-j}(A_{(\theta)}^s)(x, \xi/|\xi|)|_{s=0}, \quad \text{for } j \geq 0. \quad (2.18)$$

From (2.5) and (2.17) we immediately obtain the following

Lemma 2.9. *Let $B \in \operatorname{CL}^m(M, E)$ be a classical elliptic pseudo-differential operator of positive order and suppose that θ is an Agmon angle for B . Then for any logarithmic pseudo-differential operator $A \in \widetilde{\operatorname{CL}}^{n,m}(M, E)$ the difference $A - \log_{(\theta)} B \in \operatorname{CL}^n(M, E)$. In particular, any $A \in \widetilde{\operatorname{CL}}(M, E)$ can be represented in the form*

$$A = A_1 + \log_{(\theta)} A_2, \quad (2.19)$$

where A_1, A_2 are classical pseudo-differential operators.

2.10. Dependence of logarithm on the angle. Let $\Pi_{(\theta_1, \theta_2)}$ denote the spectral projection of A corresponding to the eigenvalues which lie in the solid angle

$$\Lambda_{(\theta_1, \theta_2)} := \{ \rho e^{i\alpha} : \rho \geq 0, \theta_1 < \alpha < \theta_2 \}.$$

In particular, $\Pi_{(\theta_1, \theta_2)} = \text{Id}$ if $|\theta_2 - \theta_1| \geq 2\pi$.

Suppose now that θ_1, θ_2 are Agmon angles for A . It follows immediately from the definition (2.16) of the logarithm that if $2k\pi \leq \theta_2 - \theta_1 < 2(k+1)\pi$ for some integer k then, cf. formula (1.4) of [16],

$$\log_{(\theta_2)} A = 2ik\pi \text{Id} + 2i\pi \Pi_{\theta_1, \theta_2 - 2k\pi} + \log_{(\theta_1)} A. \quad (2.20)$$

In particular, if both, θ and $\theta - 2k\pi$ ($k \in \mathbb{Z}$) are Agmon angles for A , then

$$\log_{(\theta)} A = 2ik\pi \text{Id} + \log_{(\theta - 2k\pi)} A. \quad (2.21)$$

If $A \in \text{CL}(M, E)$ is a classical pseudo-differential operator we denote by $\text{Res } A$ the non-commutative residue of Wodzicki [22, 23] and Guillemin [10] (see also [11] for a review).

Lemma 2.11. *Suppose $A_j \in \text{CL}^{m_j}(M, E)$ ($j = 1, 2$) are classical elliptic pseudo-differential operators of positive orders. Then the number*

$$\text{Res} \left(\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2} \right)$$

is independent of the choice of the Agmon angles θ_1 and θ_2 .

Note that the operator $\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2}$ is classical and hence its residue is well defined.

Proof. Let $\theta'_j \in [2k_j\pi + \theta_j, 2(k_j+1)\pi + \theta_j]$ be another Agmon angle for A_j ($j = 1, 2$). Let $\Pi_{\theta_j, \theta'_j - 2k_j\pi}(A_j)$ denote the spectral projection of A_j corresponding to the solid angle $\Lambda_{\theta_j, \theta'_j - 2k_j\pi}$. Then it follows from (2.20) that

$$\begin{aligned} & \left(\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2} \right) - \left(\frac{\log_{(\theta'_1)} A_1}{m_1} - \frac{\log_{(\theta'_2)} A_2}{m_2} \right) \\ &= 2i\pi \left(\frac{k_1}{m_1} - \frac{k_2}{m_2} \right) \text{Id} + \frac{2i\pi}{m_1} \Pi_{\theta_1, \theta'_1 - 2k_1\pi}(A_1) - \frac{2i\pi}{m_2} \Pi_{\theta_2, \theta'_2 - 2k_2\pi}(A_2). \end{aligned} \quad (2.22)$$

Since the Wodzicki residue of a pseudo-differential projection vanishes, cf. [22, §6], [23], the residue of the right hand side of (2.22) is equal to 0. \square

2.12. Logarithms of odd class elliptic pseudo-differential operators. From equality (2.18) and Lemma 2.7 we obtain the following

Lemma 2.13. *Suppose $A \in \text{CL}_{(-1)}^m(M, E)$ and both, θ and $\theta - m\pi$, are Agmon angles for A , then*

$$(\log_{(\theta)} A, im\pi \text{Id} + \log_{(\theta - m\pi)} A) \quad (2.23)$$

is an odd pair of operators, cf. Subsection 2.4.

In particular, if m is even, then $\log_{(\theta)} A \in \widetilde{\text{CL}}_{(-1)}^{0,m}(M, E)$.

Proof. Since $A_{(\theta)}^s|_{s=0}$ is the identity operator, the positive homogeneous components of its symbol are given by

$$\sigma_{sm-j}(A_{(\theta)}^s)(x, \xi)|_{s=0} = \delta_{j,0} \text{Id}. \quad (2.24)$$

Thus, from (2.15) we obtain

$$\partial_s \sigma_{sm-j}(A_{(\theta)}^s)(x, -\xi/|\xi|)|_{s=0} = im\pi \delta_{j,0} + (-1)^j \partial_s \sigma_{sm-j}(A_{(\theta-m\pi)}^s)(x, \xi/|\xi|)|_{s=0}. \quad (2.25)$$

Hence, by (2.18), the pair (2.23) is odd.

If m is even, then from (2.21) and (2.23) we conclude now that the pair $(\log_{(\theta)} A, \log_{(\theta)} A)$ is odd, which, by definition of the odd pair, means that $\log_{(\theta)} A \in \widetilde{\text{CL}}_{(-1)}^{0,m}(M, E)$. \square

Corollary 2.14. *Let $A \in \text{CL}_{(-1)}^m(M, E)$ and both, θ and $\theta - m\pi$, are Agmon angles for A . Then*

$$\log_{(\theta)} A + \log_{(\theta-m\pi)} A \in \widetilde{\text{CL}}_{(-1)}^{0,m}(M, E).$$

Proposition 2.15. *Suppose $A_j \in \text{CL}_{(-1)}^{m_j}(M, E)$ ($j = 1, 2$) are odd class classical elliptic pseudo-differential operators of positive orders on an odd dimensional manifold M . Then*

$$\text{Res} \left(\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2} \right) = 0. \quad (2.26)$$

for any Agmon angles θ_1 and θ_2 .

Proof. By Lemma 2.11 the left hand side of (2.26) is independent of θ_1, θ_2 . Hence, we can assume that θ_1, θ_2 are chosen so that the angles $\theta_1 - m_1\pi, \theta_2 - m_2\pi$ are also Agmon angles for operators A_1, A_2 respectively. From Corollary 2.14 we see that

$$\left(\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2} \right) + \left(\frac{\log_{(\theta_1-m_1\pi)} A_1}{m_1} - \frac{\log_{(\theta_2-m_2\pi)} A_2}{m_2} \right) \in \text{CL}_{(-1)}^0(M, E).$$

Since the Wodzicki residue of an odd class operator on an odd dimensional manifold vanishes, cf. [13, Lemma 7.3], [12, Remark 4.5], it follows that

$$\text{Res} \left(\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2} \right) = -\text{Res} \left(\frac{\log_{(\theta_1-m_1\pi)} A_1}{m_1} - \frac{\log_{(\theta_2-m_2\pi)} A_2}{m_2} \right). \quad (2.27)$$

On the other side, by Lemma 2.11

$$\text{Res} \left(\frac{\log_{(\theta_1)} A_1}{m_1} - \frac{\log_{(\theta_2)} A_2}{m_2} \right) = \text{Res} \left(\frac{\log_{(\theta_1-m_1\pi)} A_1}{m_1} - \frac{\log_{(\theta_2-m_2\pi)} A_2}{m_2} \right). \quad (2.28)$$

The equality (2.26) follows immediately from (2.27) and (2.28). \square

3. THE SYMMETRIZED TRACE

In the first part of this section we recall the notion of *weighted trace* of a pseudo-differential operator and discuss its basic properties, cf. [15, 8, 9]. Then we define a new canonical trace, called the *symmetrized trace* Tr^{sym} , of a logarithmic pseudo-differential operator of *odd class* on an *odd dimensional manifold* and show that it is *tracial*, in the sense that $\text{Tr}^{\text{sym}}[A, B] = 0$ for all $A, B \in \widetilde{\text{CL}}_{(-1)}(M, E)$. For odd class *classical* pseudo-differential operators the symmetrized trace

coincides with the Kontsevich-Vishik canonical trace $\text{Tr}_{(-1)}$ introduced in [12, 13] by a slightly different procedure. Thus our symmetrized trace extends the Kontsevich-Vishik canonical trace to logarithmic pseudo-differential operators of odd class.

3.1. Weighted traces. Let $Q \in \text{CL}(M, E)$ be a classical elliptic pseudo-differential operator of positive order admitting an Agmon angle θ . For $A \in \widehat{\text{CL}}(M, E)$, consider the generalized ζ -function

$$s \mapsto \zeta_\theta(s, Q; A) := \text{TR}(AQ_{(\theta)}^s), \quad (3.1)$$

where TR stands for the Kontsevich-Vishik canonical trace, introduced in [13, §3], [12, §3] for classical pseudo-differential operators, and generalized in [14, §5]. For $\text{Re } s \ll -1$, the operator $AQ_{(\theta)}^s$ is of trace class and the Kontsevich-Vishik trace coincides with the usual trace.

Lemma 3.2. *The function (3.1) is meromorphic in s and has at most simple pole at $s = 0$.*

Proof. For $A \in \text{CL}(M, E)$ it is shown in [13, Th. 3.1], [12, Prop. 3.4]. For $A = \log_{(\theta)} B$ the statement of the lemma is proven in Prop. 2 of [9]. The general case, follows from this two special cases and the existence of the decomposition (2.19). \square

We shall use the following notations: Suppose $f(s)$ is a function of a complex parameter s which is meromorphic near $s = 0$. We call the zero order term in the Laurent expansion of f near $s = 0$ the *finite part of f at 0* and denote it by $\text{F.p.}_{s=0} f(s)$.

Definition 3.3. *Let $Q \in \text{CL}(M, E)$ be a classical elliptic pseudo-differential operator of positive order admitting an Agmon angle θ . The Q -weighted trace $\text{Tr}^Q A$ of a logarithmic pseudo-differential operator $A \in \widehat{\text{CL}}(M, E)$ is defined as*

$$\text{Tr}_{(\theta)}^Q A := \text{F.p.}_{s=0} \text{TR}(AQ_{(\theta)}^s). \quad (3.2)$$

A slightly more explicit expression for $\text{Tr}_{(\theta)}^Q A$ is obtained in Definitions 3 and 4 of [9].

Note that though traditionally we call (3.2) a trace it is not a trace on the algebra of pseudo-differential operators since it does not vanishes on commutators. More precisely, Proposition 1 of [8] states that

Proposition 3.4. *Suppose $Q \in \text{CL}^m(M, E)$ is a classical elliptic pseudo-differential operator of positive order admitting an Agmon angle θ . Then for any classical pseudo-differential operator $A, B \in \text{CL}(M, E)$*

$$\text{Tr}_{(\theta)}^Q [A, B] = -\frac{1}{m} \text{Res} \left([\log_{(\theta)} Q, A] B \right), \quad (3.3)$$

where $[\cdot, \cdot]$ denotes the commutator of operators and Res stands for the non-commutative residue of Wodzicki [22, 23] and Guillemin [10] (see also [11] for a review).

Note, cf. [8, Prop. 1(i)], that $[\log_{(\theta)} Q, A]$ and, hence, $[\log_{(\theta)} Q, A] B$ are classical pseudo-differential operators. Therefore the non-commutative residue in the right hand side of (3.3) is well defined.

The following proposition (cf. Lemma 0.1 of [17]) describes the dependence of the weighted trace on the operator Q .

Proposition 3.5. *Let $Q_1 \in \text{CL}^{m_1}(M, E)$, $Q_2 \in \text{CL}^{m_2}(M, E)$ be positive order classical elliptic pseudo-differential operators with Agmon angles θ_1 and θ_2 respectively. For any $A \in \widetilde{\text{CL}}^{n, \gamma}(M, E)$,*

$$\text{Tr}_{(\theta_1)}^{Q_1} A - \text{Tr}_{(\theta_2)}^{Q_2} A = \text{Res } W, \quad (3.4)$$

where

$$W := \left(\frac{\gamma \log_{(\theta_1)} Q_1}{2m_1} + \frac{\gamma \log_{(\theta_2)} Q_2}{2m_2} - A \right) \cdot \left(\frac{\log_{(\theta_1)} Q_1}{m_1} - \frac{\log_{(\theta_2)} Q_2}{m_2} \right). \quad (3.5)$$

It follows immediately from (2.17) that the operators

$$\frac{\gamma \log_{(\theta_1)} Q_1}{2m_1} + \frac{\gamma \log_{(\theta_2)} Q_2}{2m_2} - A \quad \text{and} \quad \frac{\log_{(\theta_1)} Q_1}{m_1} - \frac{\log_{(\theta_2)} Q_2}{m_2}$$

are classical. Hence, so is W . Thus the non-commutative residue in the right hand side of (3.4) is well defined.

Note that if $A \in \text{CL}^n(M, E)$ is a classical pseudo-differential operator, then $\gamma = 0$ and (3.4) reduces to

$$\text{Tr}_{(\theta_1)}^{Q_1} A - \text{Tr}_{(\theta_2)}^{Q_2} A = - \text{Res} \left[A \cdot \left(\frac{\log_{(\theta_1)} Q_1}{m_1} - \frac{\log_{(\theta_2)} Q_2}{m_2} \right) \right]. \quad (3.6)$$

3.6. The symmetrized trace. Let $Q \in \text{CL}_{(-1)}^m(M, E)$, where m is a positive integer, and suppose that both, θ and $\theta - m\pi$, are Agmon angles for Q . For $A \in \widetilde{\text{CL}}_{(-1)}(M, E)$ consider the symmetrized Q -trace

$$\text{Tr}_{(\theta)}^{Q, \text{sym}} A := \frac{1}{2} \left(\text{Tr}_{(\theta)}^Q A + \text{Tr}_{(\theta - m\pi)}^Q A \right). \quad (3.7)$$

Proposition 3.7. *Let $Q_j \in \text{CL}_{(-1)}^{m_j}(M, E)$ ($j = 1, 2$) be odd class classical elliptic pseudo-differential operators and suppose that θ_j and $\theta_j - m_j\pi$ are Agmon angles for Q_j ($j = 1, 2$). Then, for any odd class logarithmic operator $A \in \widetilde{\text{CL}}_{(-1)}^{n, \gamma}(M, E)$,*

$$\text{Tr}_{(\theta_1)}^{Q_1, \text{sym}} A = \text{Tr}_{(\theta_2)}^{Q_2, \text{sym}} A. \quad (3.8)$$

Proof. Set

$$\begin{aligned} S_{\theta_1, \theta_2} &:= \frac{\gamma \log_{(\theta_1)} Q_1}{2m_1} + \frac{\gamma \log_{(\theta_2)} Q_2}{2m_2} - A, \\ T_{\theta_1, \theta_2} &:= \frac{\log_{(\theta_1)} Q_1}{m_1} - \frac{\log_{(\theta_2)} Q_2}{m_2}. \end{aligned}$$

By Proposition 3.5,

$$\text{Tr}_{(\theta_1)}^{Q_1, \text{sym}} A - \text{Tr}_{(\theta_2)}^{Q_2, \text{sym}} A = \frac{1}{2} \text{Res} \left(S_{\theta_1, \theta_2} \cdot T_{\theta_1, \theta_2} + S_{\theta_1 - m_1\pi, \theta_2 - m_2\pi} \cdot T_{\theta_1 - m_1\pi, \theta_2 - m_2\pi} \right) \quad (3.9)$$

From Lemma 2.13 we conclude that $(S_{\theta_1, \theta_2}, S_{\theta_1 - m_1\pi, \theta_2 - m_2\pi} + i\gamma\pi)$ and $(T_{\theta_1, \theta_2}, T_{\theta_1 - m_1\pi, \theta_2 - m_2\pi})$ are odd pairs of operators. Hence, by Lemma 2.5, the pair

$$(S_{\theta_1, \theta_2} \cdot T_{\theta_1, \theta_2}, S_{\theta_1 - m_1\pi, \theta_2 - m_2\pi} \cdot T_{\theta_1 - m_1\pi, \theta_2 - m_2\pi} + i\gamma\pi T_{\theta_1 - m_1\pi, \theta_2 - m_2\pi})$$

is also odd. Therefore,

$$S_{\theta_1, \theta_2} \cdot T_{\theta_1, \theta_2} + S_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi} \cdot T_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi} + i\gamma\pi T_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi}$$

is an odd class classical pseudo-differential operator. Since the Wodzicki residue of an odd class operator on an odd dimensional manifold vanishes, cf. [13, Lemma 7.3], [12, Remark 4.5], it follows that

$$\text{Res} \left(S_{\theta_1, \theta_2} \cdot T_{\theta_1, \theta_2} + S_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi} \cdot T_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi} \right) = -i\gamma\pi \text{Res} T_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi}. \quad (3.10)$$

By Proposition 2.15, $\text{Res} T_{\theta_1 - m_1 \pi, \theta_2 - m_2 \pi} = 0$. Hence (3.8) follows from (3.10) and (3.9) \square

Proposition 3.7 justifies the following

Definition 3.8. Let $A \in \widetilde{\text{CL}}_{(-1)}(M, E)$ be an odd class logarithmic operator on an odd-dimensional closed manifold M . The symmetrized trace $\text{Tr}^{\text{sym}} A$ of A is defined by the formula

$$\text{Tr}^{\text{sym}} A := \text{Tr}_{(\theta)}^{Q, \text{sym}} A,$$

where $Q \in \text{CL}_{(-1)}^m(M, E)$ is any odd class classical elliptic pseudo-differential operator of positive order m , which admits an Agmon angle, and $\theta \in [0, 2\pi)$ is an arbitrary Agmon angle for Q such that $\theta - m\pi$ is also an Agmon angle for Q .

The main property of the symmetrized trace is that it is tracial, i.e., vanishes on commutators. More precisely, the following theorem holds.

Theorem 3.9. Suppose M is an odd-dimensional closed manifold and let E be a vector bundle over M . Then

$$\text{Tr}^{\text{sym}}[A, B] = 0 \quad (3.11)$$

for any operators $A, B \in \widetilde{\text{CL}}_{(-1)}(M, E)$.

Proof. Let $Q \in \text{CL}_{(-1)}^m(M, E)$ be an odd class classical elliptic pseudo-differential operator of a positive order m , and assume that both, θ and $\theta - m\pi$, are Agmon angles for Q .

Consider first the case when both, A and B , are classical. Let θ and $\theta - m\pi$ be Agmon angles for Q . By Proposition 3.4,

$$\text{Tr}^{Q, \text{sym}}[A, B] = -\frac{1}{2m} \text{Res} \left([\log_{(\theta)} Q + \log_{(\theta - m\pi)} Q, A] B \right). \quad (3.12)$$

By Corollary 2.14, $\log_{(\theta)} Q + \log_{(\theta - m\pi)} Q \in \widetilde{\text{CL}}_{(-1)}(M, E)$. Hence, by Lemma 2.3(ii),

$$[\log_{(\theta)} Q + \log_{(\theta - m\pi)} Q, A] B \in \text{CL}_{(-1)}(M, E).$$

Since the Wodzicki residue of an odd class operator on an odd dimensional manifold vanishes, cf. [13, Lemma 7.3], [12, Remark 4.5], equality (3.11) for the case $A, B \in \text{CL}_{(-1)}(M, E)$ follows from (3.12).

Assume now that $A \in \widetilde{\text{CL}}_{(-1)}^{m_1, \gamma_1}(M, E)$, $B \in \widetilde{\text{CL}}_{(-1)}^{m_2, \gamma_2}(M, E)$. Then the operators $A - \frac{\gamma_1}{m} \log_{(\theta)} Q$ and $B - \frac{\gamma_2}{m} \log_{(\theta)} Q$ are classical. Hence, it follows from the equality

$$[A, B] = \left[A - \frac{\gamma_1}{m} \log_{(\theta)} Q, B - \frac{\gamma_2}{m} \log_{(\theta)} Q \right] + \left[\log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right]$$

that

$$\mathrm{Tr}^{\mathrm{sym}}[A, B] = \mathrm{Tr}^{\mathrm{sym}} \left[\log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right]. \quad (3.13)$$

If θ' is any Agmon angle for Q , then the operators $Q_{(\theta')}^s$ and $\log_{(\theta)} Q$ commute. Therefore, for every $s \in \mathbb{C} \setminus \mathbb{Z}$,

$$\mathrm{TR} Q_{(\theta')}^s \left[\log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right] = \mathrm{TR} \left[Q_{(\theta')}^s \log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right] = 0.$$

Hence,

$$\mathrm{Tr}_{(\theta')}^Q \left[\log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right] = \mathrm{F.p.}_{s=0} \mathrm{TR} Q_{(\theta')}^s \left[\log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right] = 0.$$

From the definition (3.7) of the symmetrized trace we now conclude that

$$\mathrm{Tr}^{\mathrm{sym}} \left[\log_{(\theta)} Q, \frac{\gamma_2}{m} A - \frac{\gamma_1}{m} B \right] = 0,$$

which in view of (3.13) implies (3.11). \square

3.10. The Kontsevich-Vishik canonical trace on the algebra of odd class classical pseudo-differential operators. For the case when $A \in \mathrm{CL}_{(-1)}(M, E)$ is an odd class classical pseudo-differential operator Kontsevich and Vishik [12, §4] defined a canonical trace $\mathrm{Tr}_{(-1)} A$ as follows: Consider a classical elliptic positive definite operator Q . Then, cf. [12, Prop. 4.1], the function $s \mapsto \mathrm{TR} A Q_{(\pi)}^s$ is regular at 0 and

$$\mathrm{Tr}_{(-1)} A := \mathrm{TR} A Q_{(\pi)}^s \Big|_{s=0} \quad (3.14)$$

is independent of the choice of Q . Note that such a Q must have an even order. If we assume that Q is an odd class operator, then by (3.7) we obtain

$$\mathrm{Tr}_{(-1)} A := \mathrm{Tr}_{(\pi)}^{Q, \mathrm{sym}} A = \mathrm{Tr}^{\mathrm{sym}} A. \quad (3.15)$$

Thus, in the case when A is classical, our symmetrized trace coincides with $\mathrm{Tr}_{(-1)}$. The advantage of formula (3.15) is that it allows to compute the canonical trace using an operator Q which is not positive definite. For example, using the operator Q whose order is odd. This will be important for applications to determinants in the next section.

4. THE SYMMETRIZED DETERMINANT

In this section we define the *symmetrized determinant* of an odd class elliptic operator on an odd-dimensional manifold M . For operators of even order the symmetrized determinant coincides with the usual ζ -regularized determinant, but for operators of odd order it might be quite different. In the next section we show that up to sign the symmetrized determinant is very often multiplicative. This result generalizes the theorem of Kontsevich and Vishik [13, Th. 7.1], [12, Th. 4.1], that on odd dimensional manifold the ζ -regularized determinant is multiplicative when restricted to odd class operators, which are close to positive definite ones.

4.1. Definition of the symmetrized determinant. Recall, cf., for example, [9, §5], that the ζ -regularized determinant of an elliptic classical pseudo-differential operator $A \in \text{CL}(M, E)$ with an Agmon angle θ can be defined by the formula

$$\log \text{Det}_{(\theta)} A = \text{Tr}_{(\theta)}^A \log_{(\theta)} A = \left. \frac{d}{ds} \right|_{s=0} \text{TR } A_{(\theta)}^s. \quad (4.1)$$

Note that (4.1) defines a particular branch of the logarithm of $\text{Det}_{(\theta)} A$.

We now define a symmetrized version of the ζ -regularized determinant as follows:

Definition 4.2. *Let M be an odd dimensional closed manifold and let E be a vector bundle over M . Suppose $A \in \text{CL}_{(-1)}^m(M, E)$ is an odd class classical elliptic pseudo-differential operator of positive order m and let both, θ and $\theta - m\pi$, be Agmon angles for A . The logarithm of the symmetrized determinant $\text{Det}_{(\theta)}^{\text{sym}} A$ is defined by the formula²*

$$\log \text{Det}_{(\theta)}^{\text{sym}} A := \frac{1}{2} \text{Tr}^{\text{sym}} (\log_{(\theta)} A + \log_{(\theta-m\pi)} A). \quad (4.2)$$

We now give an alternative formula for the symmetrized determinant.

Proposition 4.3. *Under the assumptions of Definition 4.2 we have*

$$\log \text{Det}_{(\theta)}^{\text{sym}} A := \frac{1}{2} \left(\log \text{Det}_{(\theta)} A + \log \text{Det}_{(\theta-m\pi)} A \right). \quad (4.3)$$

Proof. Clearly, (4.3) is equivalent to

$$\text{Tr}_{(\theta)}^A \log_{(\theta-m\pi)} A + \text{Tr}_{(\theta-m\pi)}^A \log_{(\theta)} A = \text{Tr}_{(\theta)}^A \log_{(\theta)} A + \text{Tr}_{(\theta-m\pi)}^A \log_{(\theta-m\pi)} A,$$

and, hence, to

$$\text{Tr}_{(\theta)}^A [\log_{(\theta)} A - \log_{(\theta-m\pi)} A] - \text{Tr}_{(\theta-m\pi)}^A [\log_{(\theta)} A - \log_{(\theta-m\pi)} A] = 0. \quad (4.4)$$

By (2.20),

$$\log_{(\theta)} A - \log_{(\theta-m\pi)} A = 2ik\pi \text{Id}, \quad (4.5)$$

if $m = 2k$ is even, and

$$\log_{(\theta)} A - \log_{(\theta-m\pi)} A = 2ik\pi \text{Id} + 2i\pi \Pi_{\theta-\pi, \theta}, \quad (4.6)$$

if $m = 2k + 1$ is odd. In both cases, $\log_{(\theta)} A - \log_{(\theta-m\pi)} A$ is a classical pseudo-differential operator. Hence, it follows from (3.6) that the left hand side of (4.4) is equal to

$$-\frac{1}{m} \text{Res} \left[(\log_{(\theta)} A - \log_{(\theta-m\pi)} A)^2 \right]. \quad (4.7)$$

Since the Wodzicki residue of a pseudo-differential projection vanishes, cf. [22, §6], [23], we have

$$\text{Res Id} = \text{Res } \Pi_{\theta-\pi, \theta} = 0.$$

From (4.5) and (4.6) we conclude now that (4.7) is equal to 0 and, hence, (4.4) holds. \square

Proposition 4.4. *If $m = 2k$ is even then $\text{Det}_{(\theta)}^{\text{sym}} A = \text{Det}_{(\theta)} A$.*

²By Corollary 2.14, $\log_{(\theta)} A + \log_{(\theta-m\pi)} A$ is an odd class logarithmic pseudo-differential operator. Hence, the symmetrized trace in the right hand side of (4.3) is well defined.

Proof. As m is even,

$$A_{(\theta-m\pi)}^s = e^{-im\pi s} A_{(\theta)}^s.$$

Hence, from (4.1) we obtain

$$\log \text{Det}_{(\theta-m\pi)} A = \log \text{Det}_{(\theta)} A - im\pi \text{TR} A_{(\theta)}^s \Big|_{s=0}.$$

Since the ζ -function of an odd class elliptic operator of even order on odd dimensional manifold vanishes at 0, cf. [20], [21, §II.13], [7, Th. 1.1], we conclude that $\log \text{Det}_{(\theta-m\pi)} A = \log \text{Det}_{(\theta)} A$. The proposition follows now from (4.3). \square

Proposition 4.5. *If $m = 2k + 1$ is odd then*

$$\text{Det}_{(2\theta)}^{\text{sym}}(A^2) = (\text{Det}_{(\theta)}^{\text{sym}} A)^2. \quad (4.8)$$

Proof. Let Π_+ and Π_- denote the spectral projections of A corresponding to the solid angles $\Lambda_{(\theta-\pi, \theta)}$ and $\Lambda_{(\theta-2\pi, \theta-\pi)}$ respectively. Then $\Pi_+ + \Pi_- = \text{Id}$. Clearly,

$$\begin{aligned} \text{TR} (A^2)_{(2\theta)}^{s/2} &= \text{TR} [\Pi_+ A_{(\theta)}^s] + \text{TR} [\Pi_- (-A)_{(\theta)}^s]; \\ \text{TR} A_{(\theta)}^s &= \text{TR} [\Pi_+ A_{(\theta)}^s] + e^{-i\pi s} \text{TR} [\Pi_- (-A)_{(\theta)}^s] \\ &= \text{TR} (A^2)_{(2\theta)}^{s/2} + (e^{-i\pi s} - 1) \text{TR} [\Pi_- (-A)_{(\theta)}^s]; \end{aligned}$$

and

$$\begin{aligned} \text{TR} A_{(\theta-m\pi)}^s &= e^{-i\pi(m+1)s} \text{TR} [\Pi_+ A_{(\theta)}^s] + e^{-i\pi ms} \text{TR} [\Pi_- (-A)_{(\theta)}^s] \\ &= e^{-i\pi(m+1)s} \text{TR} (A^2)_{(2\theta)}^{s/2} + (e^{-i\pi ms} - e^{-i\pi(m+1)s}) \text{TR} [\Pi_- (-A)_{(\theta)}^s]. \end{aligned}$$

Since the Wodzicki residue of a pseudo-differential projection vanishes, cf. [22, §6], [23], the functions $\text{TR} [\Pi_+ A_{(\theta)}^s]$ and $\text{TR} [\Pi_- (-A)_{(\theta)}^s]$ are regular at 0. Hence,

$$\frac{d}{ds} \Big|_{s=0} \text{TR} A_{(\theta)}^s = \frac{d}{ds} \Big|_{s=0} \text{TR} (A^2)_{(2\theta)}^{s/2} - i\pi \text{TR} [\Pi_- (-A)_{(\theta)}^s] \Big|_{s=0}. \quad (4.9)$$

Similarly,

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \text{TR} A_{(\theta-m\pi)}^s &= \frac{d}{ds} \Big|_{s=0} \text{TR} (A^2)_{(2\theta)}^{s/2} \\ &\quad - i(m+1)\pi \text{TR} (A^2)_{(2\theta)}^{s/2} \Big|_{s=0} + i\pi \text{TR} [\Pi_- (-A)_{(\theta)}^s] \Big|_{s=0}. \end{aligned} \quad (4.10)$$

Since the ζ -function of an odd class elliptic operator of even order on odd dimensional manifold vanishes at 0, cf. [20], [21, §II.13], [7, Th. 1.1], we conclude from (4.10) that

$$\frac{d}{ds} \Big|_{s=0} \text{TR} A_{(\theta-m\pi)}^s = \frac{d}{ds} \Big|_{s=0} \text{TR} (A^2)_{(2\theta)}^{s/2} + i\pi \text{TR} [\Pi_- (-A)_{(\theta)}^s] \Big|_{s=0}. \quad (4.11)$$

From (4.9), (4.11) and (4.3) we obtain (4.8). \square

4.6. Dependence on the angle. For the usual ζ -regularized determinant $\text{Det}_{(\theta)} A$, we have $\text{Det}_{(\theta)} A = \text{Det}_{(\theta')} A$ if there are only finitely many eigenvalues of A in the solid angle

$$\Lambda_{(\theta, \theta')} := \{ \rho e^{i\alpha} : \rho \geq 0, \theta \leq \alpha \leq \theta' \}.$$

If the order m of A is even, then in view of Proposition 4.4 the same is true for the symmetrized determinant $\text{Det}_{(\theta)}^{\text{sym}} A$. However, if m is odd, then the above condition does not imply that $\text{Det}_{(\theta)}^{\text{sym}} A = \text{Det}_{(\theta')}^{\text{sym}} A$ since it might happen that there are infinitely many eigenvalues of A in the solid angle $\Lambda_{(\theta-\pi, \theta'-\pi)}$. Moreover, because of the factor $1/2$ in the definition (4.2) even if there are only finitely many eigenvalues of A in the solid angles $\Lambda_{(\theta, \theta')}$ and $\Lambda_{(\theta-\pi, \theta'-\pi)}$, we only have

$$\text{Det}_{(\theta)}^{\text{sym}} A = \pm \text{Det}_{(\theta')}^{\text{sym}} A. \quad (4.12)$$

If there are infinitely many eigenvalues in $\Lambda_{(\theta, \theta')}$, then (4.12) does not hold in general. However, the dependence of $\text{Det}_{(\theta)} A$ on θ in this case is weaker than the dependence of the usual ζ -regularized determinant. In particular, the following proposition holds.

Proposition 4.7. *Let M be an odd-dimensional closed manifold and let E be a vector bundle over M . Suppose $A \in \text{CL}_{(-1)}^m(M, E)$ is an elliptic pseudo-differential operator of odd positive order $m = 2l - 1$ and that θ_i and $\theta_i - \pi$ ($i = 1, 2$) are Agmon angles for A . Suppose that $0 \leq \theta_2 - \theta_1 < \pi$ and that all but finitely many eigenvalues of A lie in $\Lambda_{(\theta_1, \theta_2)} \cup \Lambda_{(\theta_1 - \pi, \theta_2 - \pi)}$. Then*

$$\text{Det}_{(\theta_1)}^{\text{sym}} A = \pm \text{Det}_{(\theta_2)}^{\text{sym}} A. \quad (4.13)$$

Proof. Since m is odd it follows from our assumptions that $\theta_1 - m\pi$ and $\theta_2 - m\pi$ are Agmon angles for A . Let Π_{θ_1, θ_2} and $\Pi_{\theta_1 - m\pi, \theta_2 - m\pi}$ be as in Subsection 2.10. Then

$$\begin{aligned} A_{(\theta_1)}^s - A_{(\theta_2)}^s &= (1 - e^{2\pi i s}) \Pi_{\theta_1, \theta_2} A_{(\theta_1)}^s; \\ A_{(\theta_1 - m\pi)}^s - A_{(\theta_2 - m\pi)}^s &= (1 - e^{2\pi i s}) \Pi_{\theta_1 - m\pi, \theta_2 - m\pi} A_{(\theta_1 - m\pi)}^s. \end{aligned} \quad (4.14)$$

Since the Wodzicki residue of a pseudo-differential projection vanishes, cf. [22, §6], [23], the functions

$$s \mapsto \text{TR} [\Pi_{\theta_1, \theta_2} A_{(\theta_1)}^s] \quad \text{and} \quad s \mapsto \text{TR} [\Pi_{\theta_1 - m\pi, \theta_2 - m\pi} A_{(\theta_1 - m\pi)}^s]$$

are regular at $s = 0$. Hence, from (4.1), (4.2), and (4.14), we obtain

$$\log \text{Det}_{(\theta_1)}^{\text{sym}} A - \log \text{Det}_{(\theta_2)}^{\text{sym}} A = -\pi i \text{TR} \left[\Pi_{\theta_1, \theta_2} A_{(\theta_1)}^s + \Pi_{\theta_1 - m\pi, \theta_2 - m\pi} A_{(\theta_1 - m\pi)}^s \right] \Big|_{s=0}. \quad (4.15)$$

Let $\tilde{\Pi}$ denote the spectral projection of the operator A^2 corresponding to the eigenvalues of A^2 which lie in the solid angle $\Lambda_{2\theta_1, 2\theta_2}$. Then

$$\tilde{\Pi} = \Pi_{\theta_1, \theta_2} + \Pi_{\theta_1 - m\pi, \theta_2 - m\pi} \quad (4.16)$$

and

$$\text{TR} [\Pi_{\theta_1, \theta_2} A_{(\theta_1)}^s] + e^{m\pi i s} \text{TR} [\Pi_{\theta_1 - m\pi, \theta_2 - m\pi} A_{(\theta_1 - m\pi)}^s] = \text{TR} [\tilde{\Pi} (A^2)_{(2\theta_1)}^{s/2}].$$

Hence, at $s = 0$ we obtain

$$\mathrm{TR} \left[\Pi_{\theta_1, \theta_2} A_{(\theta_1)}^s + \Pi_{\theta_1 - m\pi, \theta_2 - m\pi} A_{(\theta_1 - m\pi)}^s \right] \Big|_{s=0} = \mathrm{TR} \left[\tilde{\Pi} (A^2)_{(2\theta_1)}^{s/2} \right] \Big|_{s=0}. \quad (4.17)$$

From (4.16) and the assumptions of the proposition we conclude that $I - \tilde{\Pi}$ is a finite dimensional projection. Hence,

$$\mathrm{TR} \left[(A^2)_{(2\theta_1)}^{s/2} \right] \Big|_{s=0} - \mathrm{TR} \left[\tilde{\Pi} (A^2)_{(2\theta_1)}^{s/2} \right] \Big|_{s=0} \in \mathbb{Z}.$$

Since the ζ -function of an odd class elliptic operator of even order on an odd dimensional manifold vanishes at 0, cf. [20], [21, §II.13], we have $\mathrm{TR} \left[(A^2)_{(2\theta_1)}^{s/2} \right] \Big|_{s=0} = 0$, and, hence,

$$\mathrm{TR} \left[\tilde{\Pi} (A^2)_{(2\theta_1)}^{s/2} \right] \Big|_{s=0} \in \mathbb{Z}.$$

Therefore, from (4.15) and (4.17) we obtain

$$\log \mathrm{Det}_{(\theta_1)}^{\mathrm{sym}} A - \log \mathrm{Det}_{(\theta_2)}^{\mathrm{sym}} A \in \pi i \mathbb{Z}, \quad (4.18)$$

which is equivalent to (4.13). \square

Corollary 4.8. *Let $A \in \mathrm{CL}_{(-1)}^m(M, E)$ be an elliptic operator of odd order $m = 2l - 1$ on an odd dimensional manifold M . Assume that the leading symbol of A is self-adjoint with respect to the scalar product defined by some Hermitian metric on E . Then up to sign the symmetrized determinant $\mathrm{Det}_{(\theta)}^{\mathrm{sym}} A$ is independent of the choice of the Agmon angle θ .*

Remark 4.9. Kontsevich and Vishik, [12, §4.1], suggested to define a determinant of a Dirac-type operator A as a square root of the determinant of the Laplace-type operator A^2 . Thus their definition had a sign indeterminacy. It follows from the equation (4.8) that the reduction modulo signs of the symmetrized determinant $\mathrm{Det}_{(\pi/2)}^{\mathrm{sym}} A$ coincides in the case of Dirac-type operators with the Kontsevich-Vishik determinant. Thus in the case of Dirac-type operators our symmetrized determinant provides a more direct construction of the Kontsevich-Vishik determinant and also fixes the signs in its definition.

5. THE MULTIPLICATIVE PROPERTIES OF THE SYMMETRIZED DETERMINANT

In this section we show, that under suitable assumptions the symmetrized determinant is multiplicative up to sign. Let θ be a principal angle for an elliptic operator $A \in \mathrm{CL}_{(-1)}^m(M, E)$. We say that an Agmon angle $\theta' \geq \theta$ is sufficiently close to θ if there are no eigenvalues of A in the solid angles $\Lambda_{(\theta, \theta']}$ and $\Lambda_{(\theta - m\pi, \theta' - m\pi]}$. We shall denote by $\mathrm{Det}_{(\tilde{\theta})} A$, $\mathrm{Det}_{(\tilde{\theta})}^{\mathrm{sym}} A$, $\log_{(\tilde{\theta})} A$, etc. the corresponding numbers defined using any Agmon angle $\theta' \geq \theta$ sufficiently closed to θ . Clearly, those numbers are independent of the choice of such θ' .

Theorem 5.1. *Let M be an odd dimensional manifold and let E be a vector bundle over M . Suppose $A \in \mathrm{CL}_{(-1)}^{m_A}(M, E)$ and $B \in \mathrm{CL}_{(-1)}^{m_B}(M, E)$ are odd class classical invertible pseudo-differential operators of positive integer orders. Let θ_A and θ_B be principal angles for A and B respectively. Assume further that there exists a continuous functions $\alpha : [0, 1] \rightarrow \mathbb{R}$ such*

that $\alpha(0) = \theta_B$ and for each $t \in [0, 1]$ the angle $\alpha(t)$ is principal for the operator $A_{(\tilde{\theta}_A)}^t B$. Set $\theta_B = \alpha(0)$, $\theta_{AB} = \alpha(1)$. Then

$$\text{Det}_{(\tilde{\theta}_{AB})}^{\text{sym}} AB = \pm \text{Det}_{(\tilde{\theta}_A)}^{\text{sym}} A \cdot \text{Det}_{(\tilde{\theta}_B)}^{\text{sym}} B. \quad (5.1)$$

The proof is given in Subsection 5.4.

The assumptions of Theorem 5.1 are rather restrictive. There are, however, several important situations, where these conditions are automatically satisfied. We now mention some of these situations.

Corollary 5.2. *Assume that the leading symbols of A and B are self-adjoint and that the leading symbol of A is positive definite. Then (5.1) holds with $\theta_{AB} = \theta_B$.*

Proof. Without loss of generality we can assume that $\theta_B \notin \pi\mathbb{Z}$. The leading symbol of the operator $A_{(\tilde{\theta}_A)}^t B$ is conjugate to the self-adjoint symbol

$$\sigma_{tm_A/2}(A_{(\tilde{\theta}_A)}^{t/2} B A_{(\tilde{\theta}_A)}^{t/2}).$$

Thus θ_B is a principal angle for $A_{(\tilde{\theta}_A)}^t B$. Hence, we can apply Theorem 5.1 with $\alpha(t) = \theta_B$. \square

Corollary 5.3. *Assume that the leading symbols of A and B commute with each other and that the leading symbol of A is self-adjoint. Let $\theta_A, \theta_B \in (0, 2\pi)$ be principal angles for A and B , respectively. Then equality (5.1) holds with $\theta_{AB} = \theta_A + \varepsilon\theta_B$, where $\varepsilon = 1$ if $0 < \theta_A < \pi$ and $\varepsilon = -1$ if $\pi < \theta_A < 2\pi$.*

Proof. Since the leading symbol $\sigma_{m_A}(A)$ of A is self-adjoint, all its eigenvalues lie on the real axis. Hence, all the eigenvalues of

$$\sigma_{tm_A}(A_{(\tilde{\theta}_A)}^t) = (\sigma_{m_A}(A))_{(\theta_A)}^t$$

lie on the rays $R_0 = \{r : r > 0\}$ and $R_{t\varepsilon\pi} = \{re^{-i\varepsilon\pi} : r > 0\}$. Thus, since the leading symbols of A and B commute the eigenvalues of the leading symbol

$$\sigma_{tm_A+m_B}(A_{(\tilde{\theta}_A)}^t B) = (\sigma_{m_A}(A))_{(\theta_A)}^t \cdot \sigma_{m_B}(B)$$

have the form $r\lambda$, $r\lambda e^{-i\varepsilon t}$, where λ is an eigenvalue of $\sigma_{m_B}(B)$ and $r > 0$. Therefore we can apply Theorem 5.1 with $\alpha(t) = \theta_B + t\varepsilon\theta_A$. \square

5.4. Proof of Theorem 5.1. Without loss of generality we can assume that θ_A and $\theta_A - m_A\pi$ are Agmon angles for A , and θ_B and $\theta_B - m_B\pi$ are Agmon angles for B .

The leading symbol of the operator $A_{(\theta_A)}^t B$ is given by

$$\sigma_{tm_A+m_B}(A_{(\theta_A)}^t B)(x, \xi) = (\sigma_{m_A}(A)(x, \xi))_{(\theta_A)}^t \cdot \sigma_{m_B}(B)(x, \xi).$$

Hence, from (2.15) we obtain

$$\begin{aligned} \sigma_{tm_A+m_B}(A_{(\theta_A)}^t B)(x, -\xi) &= (-1)^{m_B} e^{im_A t \pi} \sigma_{tm_A+m_B}(A_{(\theta_A - m_A \pi)}^t B)(x, \xi) \\ &= e^{i(m_A t + m_B)\pi} \sigma_{tm_A+m_B}(A_{(\theta_A - m_A \pi)}^t B)(x, \xi). \end{aligned}$$

Therefore, if $\alpha(t)$ is a principal angle for $A_{(\theta_A)}^t B$ then

$$\beta(t) := \alpha(t) - i(m_A t + m_B)\pi$$

is a principal angle for $A_{(\theta_A - m_A \pi)}^t B$.

Using the formula for multiplicative anomaly of the ζ -regularized determinant, [12, Prop. 2.1] [17, Th. 1], we conclude that

$$\begin{aligned} \log \text{Det}_{(\theta_{AB})}^{\text{sym}} AB &= \log \text{Det}_{(\theta_A)}^{\text{sym}} A - \log \text{Det}_{(\theta_B)}^{\text{sym}} B \\ &\equiv \frac{m_A}{4} \int_0^1 \text{Res} (U(t)^2 + V(t)^2) dt, \quad \text{mod } \pi\mathbb{Z}, \end{aligned} \quad (5.2)$$

where

$$U(t) = \frac{\log(\tilde{\alpha}(t)) A_{(\theta_A)}^t B}{m_A t + m_B} - \frac{\log(\theta_A) A}{m_A}, \quad (5.3)$$

and

$$V(t) = \frac{\log(\tilde{\beta}(t)) A_{(\theta_A - m_A \pi)}^t B}{m_A t + m_B} - \frac{\log(\theta_A - m_A \pi) A}{m_A}. \quad (5.4)$$

Lemma 5.5. *For each $t \in [0, 1]$, $(U(t), V(t))$ is an odd pair of operators in the sense of Subsection 2.4.*

Proof. Fix $t \in [0, 1]$ and set $m = tm_A + m_B$. From (2.15) we easily obtain

$$\sigma_{m-j}(A_{(\theta_A)}^t B)(x, -\xi) = (-1)^j e^{im\pi} \sigma_{m-j}(A_{(\theta_A - m_A \pi)}^t B)(x, \xi). \quad (5.5)$$

Set

$$R(\lambda) := (A_{(\theta_A)}^t B - \lambda)^{-1}, \quad \hat{R}(\lambda) := (A_{(\theta_A - m_A \pi)}^t B - \lambda)^{-1}.$$

Let $r(x, \xi; \lambda) \sim \sum_{j=0}^{\infty} r_{-m-j}(x, \xi; \lambda)$ and $\hat{r}(x, \xi; \lambda) \sim \sum_{j=0}^{\infty} \hat{r}_{-m-j}(x, \xi; \lambda)$ be the symbols of $R(\lambda)$ and $\hat{R}(\lambda)$ respectively. Here the asymptotic expansions are understood in the same sense as in (2.9).

From (5.5) and the standard formulae for the parametrix, cf., for example, [12, §2], we get

$$r_{-m-j}(x, -\xi; e^{im\pi} \lambda) = (-1)^j e^{im\pi} \hat{r}_{-m-j}(x, \xi; \lambda). \quad (5.6)$$

Hence, a verbatim repetition of the computation in the proof of Lemma 2.7 yields

$$\sigma_{sm-j}((A_{(\theta_A)}^t B)_{(\tilde{\alpha}(t))}^s)(x, -\xi) = (-1)^j e^{ism\pi} \sigma_{sm-j}((A_{(\theta_A - m_A \pi)}^t B)_{(\tilde{\beta}(t))}^s)(x, \xi). \quad (5.7)$$

Since $(A_{(\theta_A)}^t B)_{(\tilde{\alpha}(t))}^s|_{s=0} = (A_{(\theta_A - m_A \pi)}^t B)_{(\tilde{\beta}(t))}^s|_{s=0} = \text{Id}$, we obtain

$$\partial_s \sigma_{sm-j}((A_{(\theta_A)}^t B)_{(\tilde{\alpha}(t))}^s)(x, \xi)|_{s=0} = \delta_{j,0} \text{Id}. \quad (5.8)$$

Combining (5.7) with (5.8) we get

$$\begin{aligned} \partial_s \sigma_{sm-j}((A_{(\theta_A)}^t B)_{(\tilde{\alpha}(t))}^s)(x, -\xi/|\xi|)|_{s=0} \\ = im\pi \delta_{j,0} + \partial_s \sigma_{sm-j}((A_{(\theta_A - m_A \pi)}^t B)_{(\tilde{\beta}(t))}^s)(x, \xi/|\xi|)|_{s=0}. \end{aligned} \quad (5.9)$$

Hence, using (2.18) we conclude that

$$\left(\log_{(\tilde{\alpha}(t))} A_{(\theta_A)}^t B, \log_{(\tilde{\beta}(t))} A_{(\theta_A - m_A \pi)}^t B + i m \pi \text{Id} \right)$$

is an odd pair of operators. The statement of the lemma follows now from Lemma 2.13 and the definitions of the operators $U(t)$, $V(t)$. \square

Combining Lemma 2.5 and Lemma 5.5 we conclude that $(U(t)^2, V(t)^2)$ is an odd pair of operators. Hence, the operator $U(t)^2 + V(t)^2$ is of odd class. Since the Wodzicki residue of an odd class operator on an odd dimensional manifold vanishes, cf. [13, Lemma 7.3], [12, Remark 4.5], we obtain $\text{Res}(U(t)^2 + V(t)^2) = 0$. Therefore, (5.1) follows from (5.2). \square

6. OPERATORS WHOSE SPECTRUM IS SYMMETRIC ABOUT THE REAL AXIS

In this section we prove that the symmetrized determinant of an operator whose spectrum is symmetric about the real axis is real and we compute its sign. Note that the spectrum of every differential operator with *real coefficients* is symmetric about the real axis. Other interesting examples of operators with symmetric spectrum are discussed in [4, 2, 1].

Definition 6.1. *The spectrum of A is symmetric with respect to the real axis if the following condition holds: if λ is an eigenvalue of A , then $\bar{\lambda}$ also is an eigenvalue of A and has the same algebraic multiplicity as λ .*

We denote by P_+ and P_- the spectral projection of A corresponding to the positive and negative parts of the imaginary axis respectively. Assume that the leading symbol of A is self-adjoint with respect to some Hermitian metric on E . Then the projections P_{\pm} have finite rank. Set $m_{\pm} = \text{rank } P_{\pm}$. Then m_+ (respectively m_-) is equal to the number of the eigenvalues of A (counted with their algebraic multiplicities) on the positive (respectively negative) part of the imaginary axis. If the spectrum of A is symmetric about the real axis then $m_+ = m_-$.

Theorem 6.2. *Let $A \in \text{CL}_{(-1)}^m(M, E)$ be a classical elliptic pseudo-differential operator of positive odd order $m = 2l - 1$, whose spectrum is symmetric about the real axis and whose leading symbol is self-adjoint with respect to some Hermitian metric on E . Choose $\theta \in (\pi/2, \pi)$ such that both, θ and $\theta - m\pi$, are Agmon angles for A and there are no eigenvalues of A in the solid angles $\Lambda_{(\pi/2, \theta]}$ and $\Lambda_{(-\pi/2, \theta - \pi]}$. Then the symmetrized determinant $\text{Det}_{(\theta)}^{\text{sym}} A$ is real and its sign is equal to $(-1)^{m_+}$, i.e.*

$$\text{Det}_{(\theta)}^{\text{sym}} A = (-1)^{m_+} |\text{Det}_{(\theta)}^{\text{sym}} A|. \quad (6.1)$$

Proof. Let Π_+ and Π_- denote the spectral projections of A corresponding to the solid angles $L_{(-\pi/2, \pi/2)}$ and $L_{(\pi/2, 3\pi/2)}$ respectively. Let P_+ and P_- denote the spectral projections of A corresponding to the rays $R_{\pi/2}$ and $R_{-\pi/2}$ respectively (here we use the notation introduced in

Subsection 2.6). Set $\tilde{\Pi}_{\pm} = \Pi_{\pm} + P_{\pm}$. Since A is injective $\text{Id} = \tilde{\Pi}_{+} + \tilde{\Pi}_{-}$. Clearly,

$$\text{TR } A_{(\theta)}^s = \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + e^{-i\pi s} \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s]; \quad (6.2)$$

$$\text{TR } A_{(\theta-m\pi)}^s = e^{-i(m+1)\pi s} \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + e^{-im\pi s} \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s]; \quad (6.3)$$

$$\text{TR}(A^2)_{(2\theta)}^{s/2} = \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s]. \quad (6.4)$$

Hence,

$$\begin{aligned} \text{TR } A_{(\theta)}^s + \text{TR } A_{(\theta-m\pi)}^s \\ = (1 + e^{-i(m+1)\pi s}) \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + (e^{-i\pi s} + e^{-im\pi s}) \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s]. \end{aligned} \quad (6.5)$$

Since the functions $\text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s]$ and $\text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s]$ are regular at $s = 0$ we conclude from (6.5) that

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \text{TR } A_{(\theta)}^s + \frac{d}{ds} \Big|_{s=0} \text{TR } A_{(\theta-m\pi)}^s \\ = 2 \frac{d}{ds} \Big|_{s=0} \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + 2 \frac{d}{ds} \Big|_{s=0} \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s] \\ - i(m+1)\pi \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] - (i\pi + im\pi) \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s] \\ = 2 \frac{d}{ds} \Big|_{s=0} \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + 2 \frac{d}{ds} \Big|_{s=0} \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s] \\ - i(m+1)\pi \left[\text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s] \right] \Big|_{s=0}. \end{aligned} \quad (6.6)$$

Using (6.4) and the fact that the ζ -function of an odd class elliptic operator of even order on odd dimensional manifold vanishes at 0, cf. [20], [21, §II.13], [7, Th. 1.1], we conclude that

$$\left[\text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s] \right] \Big|_{s=0} = 0.$$

Also by formula (A.2) of [6] that the imaginary part of

$$\frac{d}{ds} \Big|_{s=0} \text{TR} [\tilde{\Pi}_{+} A_{(\theta)}^s] + \frac{d}{ds} \Big|_{s=0} \text{TR} [\tilde{\Pi}_{-} (-A)_{(\theta)}^s]$$

is equal to $-m_{+}\pi$. Hence, we conclude from (6.6) that the imaginary part of

$$\log \text{Det}_{(\theta)}^{\text{sym}} A = \frac{1}{2} \left[\frac{d}{ds} \Big|_{s=0} \text{TR } A_{(\theta)}^s + \frac{d}{ds} \Big|_{s=0} \text{TR } A_{(\theta-m\pi)}^s \right]$$

is equal to $-m_{+}\pi$. □

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